

# BPS Magnetic Monopole Bags

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## Abstract

We explore the characteristics of spherical bags made of large numbers of BPS magnetic monopoles. There are two extreme limits. In the Abelian bag,  $N$  zeros of the Higgs field are arranged in a quasiregular lattice on a sphere of radius  $R_{\text{cr}} \sim N/v$ , where  $v$  is the Higgs vacuum expectation value. The massive gauge fields of the theory are largely confined to a thin shell at this radius that separates an interior with almost vanishing magnetic and Higgs fields from an exterior region with long-range Coulomb magnetic and Higgs fields. In the other limiting case, which we term a non-Abelian bag, the  $N$  zeros of the Higgs field are all the origin, but there is again a thin shell of radius  $R_{\text{cr}}$ . In this case the region enclosed by this shell can be viewed as a large monopole core, with small Higgs field but nontrivial massive and massless gauge fields.

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## I. INTRODUCTION

There has recently been some interest in the nature of solutions containing a large number of BPS monopoles. Although we have a good understanding of the field configurations when the number of magnetic monopoles is relatively small, the situation becomes more complicated when the number increases. If the monopoles remain well separated, the solution is, with an appropriate gauge choice, given approximately by a superposition of individual BPS monopoles. However, as the number of monopoles increases, their non-Abelian cores increase in size until they begin to overlap and the individual monopoles merge together.

Bolognesi [1] has proposed that when  $N \gg 1$  BPS magnetic monopoles are as closely packed as possible in a single region, the field configuration can be characterized as a bag, of arbitrary shape, inside of which the Higgs field vanishes. Following this work, Ward [2] explored numerically a “monopole wall” on which BPS monopoles were arranged periodically on a plane. The scalar field is approximately constant on one side of the wall, but grows linearly with the distance from the wall on the other side. This planar wall configuration can be regarded as an approximation to the field configuration near the wall of a monopole bag.

However, much remains to be explored about the structure of these solutions. As long as  $N$  remains finite (as it must for a finite size monopole bag), the Higgs field, although exponentially small in the bag interior, only vanishes exactly at a finite number of points. Similarly, the charged vector fields are only precisely zero along special lines with nontrivial vorticity. By focusing on the patterns of these zeros, and using previous work [3, 4, 5, 6, 7] on solutions with Platonic symmetries as a guide, we will obtain a more detailed picture of the monopole bag and its wall. As we will see, the characteristics of the solutions can change quite dramatically depending on the location of the zeros of the Higgs field.

One limiting case occurs when most of the zeros are located on the surface of the bag, the configuration one naturally obtains by bringing many monopoles together from spatial infinity. Up to exponentially small corrections, the fields are purely Abelian both inside and outside the bag, with the non-Abelian behavior confined to the wall region; we will call these Abelian bags. For a spherical bag, a simple calculation shows that the minimum bag radius is of order  $N/v$ , where  $v$  is the expectation value of the Higgs field. At this radius, the non-Abelian cores of the monopoles overlap, and the Higgs field is approximately zero

inside the bag. This gives Bolognesi's bag. However one can also consider the case where the monopoles are arranged on a surface with larger radius. For the spherical case, it is easy to see that the scalar field is approximately constant, but nonvanishing, inside the bag. A simple generalization of the magnetic conductor picture of Bolognesi allows one to see how to deform this configuration while keeping the Higgs field constant in the interior.

Going in the opposite direction, one can move the Higgs zeros further in toward the center. This does not reduce the size of the bag, but instead changes its character. In the limiting case, where the zeros all coincide at a point, the Higgs field remains close to zero inside the bag, but the gauge field configuration becomes truly non-Abelian. In a sense, the bag can be thought of as an extended monopole core. We call this a non-Abelian bag.

A remarkable feature of these bags is that the bag wall has a lattice structure that gives the solution a polyhedral shape. For the Abelian bag, the vertices of the lattice are defined, in an obvious manner, by the positions of the Higgs zeros. This clearly cannot happen for the non-Abelian bag, since the zeros are all at the center. We will see that a regular structure emerges nevertheless, and will argue that it is an approximately hexagonal lattice.

In the next section we will review the essential features of the theory and establish our conventions. Then, in Sec. III, we discuss the general features of spherical bags, and describe the topological constraints on the zeros of the field. We also review here the properties of the Platonic monopoles. In Secs. IV and V we give more detailed pictures of the spherical Abelian and non-Abelian bags, respectively. Section VI contains some concluding remarks.

## II. BACKGROUND AND CONVENTIONS

We consider an  $SU(2)$  gauge theory with gauge field  $V_\mu^a$  and a triplet scalar field  $\phi^a$ . We choose scales so as to set the gauge coupling to unity. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a, \quad (2.1)$$

where  $D_\mu\phi^a = \partial_\mu\phi^a + \epsilon^{abc}V_\mu^b\phi^c$ , and  $G_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + \epsilon^{abc}V_\mu^b V_\nu^c$ . We are interested in BPS solutions, which obey the Bogomolny equation

$$G_{ij}^a = \epsilon_{ijk}D_k\phi^a. \quad (2.2)$$

We will find it convenient for the most part to work in an Abelian gauge where the Higgs

field has a fixed gauge orientation,

$$\phi^a = \delta^{a3} \phi. \quad (2.3)$$

The gauge field of the unbroken electromagnetic U(1) is then

$$A_\mu = V_\mu^3, \quad (2.4)$$

while the charged massive vector meson field is

$$W_\mu = \frac{1}{\sqrt{2}}(V_\mu^1 + iV_\mu^2). \quad (2.5)$$

The components of the field strength are

$$\begin{aligned} G_{\mu\nu}^1 + iG_{\mu\nu}^2 &= \sqrt{2}(\mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu) \\ G_{\mu\nu}^3 &= F_{\mu\nu} + i(W_\mu W_\nu^* - W_\nu W_\mu^*), \end{aligned} \quad (2.6)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and the electromagnetic covariant derivative  $\mathcal{D}_\mu = \partial_\mu + iA_\mu$ .

We will consider only static configurations with no electric charge, and so assume that  $A_0$  and  $W_0$  both vanish everywhere. We adopt three-vector notation  $\mathbf{A}$  and  $\mathbf{W}$  for the spatial components of these fields and write  $\mathbf{B} = \nabla \times \mathbf{A}$ . Equation (2.2) then becomes

$$0 = \mathbf{B} + i\mathbf{W} \times \mathbf{W}^* - \nabla\phi \quad (2.7)$$

$$0 = \mathcal{D} \times \mathbf{W} + i\mathbf{W}\phi. \quad (2.8)$$

The energy density is

$$\mathcal{E} = \frac{1}{2}(\mathbf{B} + i\mathbf{W} \times \mathbf{W}^*)^2 + |\mathcal{D} \times \mathbf{W}|^2 + \frac{1}{2}(\nabla\phi)^2 + |\mathbf{W}|^2\phi^2, \quad (2.9)$$

which for solutions of the Bogomolny equations becomes

$$\mathcal{E} = (\mathbf{B} + i\mathbf{W} \times \mathbf{W}^*)^2 + 2|\mathbf{W}|^2\phi^2. \quad (2.10)$$

We write the vector fields in terms of spherical coordinates, with  $\mathbf{A} \cdot d\mathbf{r} = A_r dr + A_\theta d\theta + A_\varphi d\varphi$  or, equivalently,

$$\mathbf{A} = \hat{r}A_r + \frac{\hat{\theta}}{r}A_\theta + \frac{\hat{\varphi}}{r \sin \theta}A_\varphi. \quad (2.11)$$

In particular, the vector potential for a Dirac monopole with magnetic charge  $N$  can be chosen so that  $A_r = A_\theta = 0$  and

$$A_\varphi = N(1 - \cos \theta). \quad (2.12)$$

We also define

$$W_{\pm} = (W_{\theta} \pm \frac{i}{\sin \theta} W_{\varphi}) e^{\pm i\varphi}. \quad (2.13)$$

The covariant curl of  $W$  is then

$$\begin{aligned} \mathcal{D} \times \mathbf{W} &= \frac{\hat{r}}{r^2 \sin \theta} (\partial_{\theta} W_{\varphi} + i A_{\theta} W_{\varphi} - \partial_{\varphi} W_{\theta} - i A_{\varphi} W_{\theta}) \\ &\quad + \frac{\hat{\theta}}{r \sin \theta} (\partial_{\varphi} W_r + i A_{\varphi} W_r - \partial_r W_{\varphi} - i A_r W_{\varphi}) \\ &\quad + \frac{\hat{\phi}}{r} (\partial_r W_{\theta} + i A_r W_{\theta} - \partial_{\theta} W_r - i A_{\theta} W_r). \end{aligned} \quad (2.14)$$

Its radial component is given in terms of  $W_+$  and  $W_-$  via

$$\begin{aligned} \hat{\mathbf{r}} \cdot \mathcal{D} \times \mathbf{W} &= \frac{e^{-i\varphi}}{2r^2} \left\{ -i[\partial_{\theta} + i A_{\theta}] - \frac{1}{\sin \theta} [\partial_{\varphi} + i(A_{\varphi} - 1 + \cos \theta)] \right\} W_+ \\ &\quad + \frac{e^{i\varphi}}{2r^2} \left\{ i[\partial_{\theta} + i A_{\theta}] - \frac{1}{\sin \theta} [\partial_{\varphi} + i(A_{\varphi} + 1 - \cos \theta)] \right\} W_-. \end{aligned} \quad (2.15)$$

We also note that

$$\hat{\mathbf{r}} \cdot (\mathbf{B} + i\mathbf{W} \times \mathbf{W}^*) = B_r - \frac{1}{2r^2} |W_+|^2 + \frac{1}{2r^2} |W_-|^2. \quad (2.16)$$

### III. SHELLS, ZEROS, AND PLATONIC MONOPOLES

Consider the case of  $N \gg 1$  monopoles distributed relatively evenly on a spherical shell of radius  $R$ . The typical separation between neighboring monopoles is  $d_{\text{sep}} \sim 2\pi R/\sqrt{N}$ , which we initially assume to be much larger than the monopole core radius. Outside the monopole cores, the only nontrivial fields are the Abelian electromagnetic field and the massless scalar field with magnitude  $\phi(\mathbf{x})$ . As long as we stay sufficiently far from the monopole cores and the sphere on which they lie, and provided that we take a sufficiently “coarse-grained” view of the system, we can treat this as a spherically symmetric purely Abelian configuration. Thus, the magnetic field will be

$$\mathbf{B} \approx \begin{cases} N \frac{\hat{\mathbf{r}}}{r^2}, & r \gtrsim R + a, \\ 0 & r \lesssim R - a. \end{cases} \quad (3.1)$$

where we expect  $a$ , which measures the effective thickness of the spherical shell containing the monopole cores, to be  $\sim v^{-1}$ . Outside the monopole cores the scalar field obeys  $\nabla \phi = \mathbf{B}$ ,

with  $\phi(r = \infty) = v$ . Hence,

$$\phi \approx \begin{cases} v - \frac{N}{r}, & r \gtrsim R + a, \\ v - \frac{N}{R}, & r \lesssim R - a. \end{cases} \quad (3.2)$$

We see that there is a critical radius,  $R_{\text{cr}} \sim N/v$ , for which  $\phi$  vanishes in the interior of the sphere.<sup>1</sup> Once the sphere has shrunk to  $R_{\text{cr}}$ , the monopole cores merge together to give a thin spherical wall, within which the fields are fully non-Abelian, that separates an essentially empty interior from a purely Abelian exterior. This is the Abelian monopole bag. Because  $\phi$  cannot become negative, the bag cannot be shrunk beyond this point. The bag can be deformed from a spherical shape, but if it remains spherical its radius cannot be less than  $R_{\text{cr}}$ .

For this case of a bag with critical radius, our estimate of the wall thickness needs to be modified. This thickness is determined locally by the properties of the Higgs field near the wall, where  $\phi$  is far from its vacuum value. The only dimensional scale relevant near the wall is the separation between monopoles on the wall,  $d_{\text{sep}}(R_{\text{cr}})$ , so we expect the wall thickness to be of comparable size; i.e.,  $\sim \sqrt{N} v^{-1}$ .

Although the bag radius cannot be less than  $R_{\text{cr}}$ , this does not mean that the Higgs zeros cannot be brought closer together. In fact, one can envision bringing them all to the center, giving an  $N$ -fold zero and implying that  $\phi \sim r^N$  near the origin. For large  $N$ , this means that we have a region with very small, although not quite vanishing, Higgs field. This region thus resembles an enlarged monopole core, with both the massless gauge field and the (just barely) massive gauge field being nontrivial. How large is this region? In the exterior, at large  $r$ , the analysis above tells us that  $\phi \approx v - (N/r)$ . This behavior cannot continue for  $r < N/v$ , so we again expect to find a spherical shell of radius  $R_{\text{cr}} \sim N/v$  and thickness  $\sim \sqrt{N} v^{-1}$  separating an exterior Abelian region from an interior region of almost vanishing Higgs field. However, in contrast with our previous case, this interior region is not empty, but instead contains an intrinsically non-Abelian configuration of fields. This is the non-Abelian bag.

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<sup>1</sup> Note that  $d_{\text{sep}}(R_{\text{cr}}) \sim \sqrt{N}/v$ . The monopole core radius is set by the local value of  $\phi$ , and so only becomes comparable to  $d_{\text{sep}}(R)$  when  $(R - R_{\text{cr}})/R_{\text{cr}}$  is of order  $1/\sqrt{N}$ .

In both cases, the fields in the exterior region can be expanded in terms of spherical harmonics appropriate to magnetic charge  $N$ . In particular, the massive vector field can be expanded in terms of monopole vector harmonics [8, 9]. This field falls exponentially for  $R > R_{\text{cr}}$ , with the dominant part of the exponential tail coming from the harmonics with the lowest total angular momentum,  $J = N - 1$ . One can show [10] that any linear combination of such harmonics vanishes at precisely  $2N - 2$  points on the unit sphere. These correspond to directions in which the multimonopole core region falls off faster, thus somewhat “flattening” the shape of the wall separating the exterior and interior regions and deforming it from a sphere to a polyhedron with  $F = 2N - 2$  faces.

In fact, topological considerations imply the existence of other zeros of the fields. First, recall that the relation between the magnetic charge and the winding number of the Higgs field implies that in an  $N$  monopole solution the number of zeros (with positive winding) minus the number of antizeros (with negative winding) of the Higgs field must be precisely  $N$ . When the individual monopoles are widely separated, it is clear that there are  $N$  zeros and no antizeros. However, this is not necessarily so; we will see that there are solutions with  $N$  units of magnetic charge that have  $N + 1$  Higgs zeros, with the excess compensated by an antizero at the origin.

In addition, the various components of  $\mathbf{W}$  are also required to have zeros and, in fact, lines of zeros. If the magnetic flux through a surface  $S$  is  $4\pi P$ , then any scalar field with unit electric charge must have  $2P$  zeros on  $S$ , with zeros being counted with a positive (negative) sign if the phase of the field increases (decreases) by  $2\pi$  in going around the zero in a clockwise direction, as seen by the outward normal. The same is true of the component of  $\mathbf{W}$  normal to the surface. The result is modified for the tangential components. In particular, on a sphere centered about the origin  $W_{\pm}$  have a total of  $2P \mp 2$  zeros; the additional factor in these cases can be understood by noting the extra terms in Eq. (2.15) for the covariant curl. By considering concentric spheres of arbitrary radius, we see that these zeros (and possibly antizeros) must form lines that can only terminate on the points where the magnetic charges are located; i.e., at the zeros of the Higgs field. Note that this is consistent with our previous remarks about the zeros of the vector harmonics with lowest angular momentum, because these only contribute to  $W_{+}$ .

A connection between magnetic charge and polyhedral shape was first encountered in a curious set of solutions that have the symmetries of the Platonic solids, but not the charges

that one might naively expect to be associated with these solids. Although in each case the energy profile shows a concentration of energy at the vertices of a regular polyhedron, the number of vertices is never equal to the magnetic charge. Thus, there is a charge-three tetrahedron [3, 4], a charge-four cube [3, 4], a charge-five octahedron [5], a charge-seven dodecahedron [5], and finally an icosahedron with  $N = 11$  [6]; for each of these the number of faces is equal to  $2N - 2$ .

The locations of the zeros of the Higgs field are rather curious [7, 11]. For the tetrahedron, octahedron, and, it is believed<sup>2</sup>, the icosahedron, (i.e., the solids with triangular faces) there are zeros at each of the  $N + 1$  vertices, and an antizero at the center. It is to be stressed that these should not be viewed as simply arrangements of  $N + 1$  monopoles about a central antimonopole. Such an assembly would be expected to have roughly  $N + 2$  times the mass  $M_1$  of a single monopole, whereas these solutions saturate the BPS bound and so have mass  $M_N = NM_1$ . It should also be noted that plots [7] of the Higgs field of the tetrahedron and octahedron show that  $\phi$  remains small throughout the interior of the polyhedron. For later use, we record their numbers of vertices  $V$ , edges  $E$ , and faces  $F$ :

$$\begin{aligned} V &= N + 1, \\ E &= 3N - 3, && \text{triangular faces} \\ F &= 2N - 2. \end{aligned} \tag{3.3}$$

The energy profiles of the other two solutions, the cube and the dodecahedron, also show concentrations about the vertices of the polyhedron. However, neither has Higgs zeros at these vertices. Instead, they have multiple zeros — four-fold and seven-fold, respectively — at their centers. Again, plots of the Higgs field for the cube [7] and the dodecahedron [12] show that it remains quite small throughout the interior region. For these solutions, corresponding to the duals of the previous examples,<sup>3</sup>

$$\begin{aligned} V &= 4N - 8, \\ E &= 6N - 12, && \text{dual polyhedra} \\ F &= 2N - 2. \end{aligned} \tag{3.4}$$

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<sup>2</sup> Although the existence of the  $N=11$  icosahedral monopole solution has been established, its properties have not been as fully explored as those of the other Platonic solutions.

<sup>3</sup> The tetrahedron is self-dual; note that Eqs. (3.3) and (3.4) agree when  $N = 4$ .



#### IV. SPHERICAL ABELIAN BAGS

As outlined in Sec. III, we expect to be able to obtain an Abelian monopole bag with charge  $N \gg 1$  by arranging unit monopoles on a spherical surface of critical radius. Although we cannot have a solution that is precisely spherically symmetric, we can hope to obtain approximate spherical symmetry by arranging the monopoles in as regular a fashion as possible.

We expect to have concentrations of energy at the locations of the monopoles, so that the solutions resemble polyhedra with their vertices at the zeros of the Higgs field. If we allow for the possibility of a small number of antizeros or zeros located elsewhere, this gives us  $V = N + k$  vertices. The arguments given previously suggest that the number of faces should be  $2N - 2$ . Euler's theorem then tells us that the number of edges is  $E = 3N + k - 4 = 3F/2 + k - 1$ . The smallest possible value of  $k$  is unity, in which case all of the faces are triangular and there is a single Higgs antizero, presumably at the center. This reproduces the result of Eq. (3.3), and suggests that the corresponding three Platonic solutions can be seen as the prototypes of the Abelian bag solutions.

Let us now examine these solutions more closely, focusing on the zeros of the components of the  $\mathbf{W}$  field. We begin by recalling that on a sphere enclosing  $P$  units of magnetic charge the components  $W_r$ ,  $W_+$ , and  $W_-$  of the massive gauge field must have  $2P$ ,  $2P - 2$ , and  $2P + 2$  zeros, respectively. This can be phrased more compactly by defining the helicity  $\lambda$  to be 0, 1, and  $-1$  for  $W_r$ ,  $W_+$ , and  $W_-$ , respectively; the number of zeros for a given component of  $\mathbf{W}$  is then  $2(P - \lambda)$ . By considering a series of concentric spheres, we see that these merge into lines of zeros that can only end at zeros or antizeros of the Higgs field. If we assume that the central Higgs antizero is located at the origin, the zero lines for the component of  $\mathbf{W}$  with helicity  $\lambda$  can then include:

- a)  $n_F^\lambda = (2N - 2)k_F^\lambda$  lines from the origin, through a face, to infinity,
- b)  $n_E^\lambda = (3N - 3)k_E^\lambda$  lines from the origin, through an edge, to infinity,
- c)  $n_V^\lambda = (N + 1)k_V^\lambda$  lines from a vertex to infinity,
- d)  $n_{V'}^\lambda = (N + 1)k_{V'}^\lambda$  lines from the origin to a vertex.

For the three cases of regular polyhedra, it is clear that these lines of zeros must all be radial and must be arranged symmetrically. Equation (3.3) then implies that the various  $k_a^\lambda$  are

integers and give the numbers of lines passing through an individual face, edge, or vertex. Although the polyhedra with other values of  $N$  do not have exact symmetry, we expect a similar result to hold when  $N$  is sufficiently large. We will assume this to be the case.

Consider a sphere centered at the origin and with radius large enough that it encloses all of the Higgs zeros. Because there are  $N$  units of flux flowing outward through this sphere,  $W_r$ ,  $W_+$ , and  $W_-$  must have  $2N$ ,  $(2N - 2)$ , and  $(2N + 2)$  zeros on the sphere, respectively. This implies that

$$(2N - 2)k_F^\lambda + (3N - 3)k_E^\lambda + (N + 1)k_V^\lambda = 2N - 2\lambda. \quad (4.1)$$

If we assume that the various  $k_i^\lambda$  are independent of  $N$ , this yields two conditions:

$$2k_F^\lambda + 3k_E^\lambda + k_V^\lambda = 2, \quad (4.2)$$

from the coefficients of  $N$ , and

$$-2k_F^\lambda - 3k_E^\lambda + k_V^\lambda = -2\lambda, \quad (4.3)$$

from the  $N$ -independent terms. These imply that

$$k_V^\lambda = 1 - \lambda \quad (4.4)$$

and

$$2k_F^\lambda + 3k_E^\lambda = 1 + \lambda. \quad (4.5)$$

Now consider a sphere centered at the origin, but lying inside the zeros of the Higgs field. Because this sphere encloses only the antizero at the origin, it should have  $-2$ ,  $-4$ , and  $0$  zeros of  $W_r$ ,  $W_+$ , and  $W_-$ , respectively; i.e., each component of  $W$  must have  $2N + 2$  fewer zeros on this inner sphere than on the outer one. It immediately follows that  $k_{V'}^\lambda = k_V^\lambda - 2$ , so that

$$k_{V'}^\lambda = -1 - \lambda. \quad (4.6)$$

As a consistency check, consider a sphere that encloses a single vertex, with one unit of flux flowing outward; we make take this sphere to be small enough that only lines connected to a vertex run through it. To count zeros on this sphere,  $W$  must be decomposed into components  $\tilde{W}_r$ ,  $\tilde{W}_+$ , and  $\tilde{W}_-$  defined with respect to its center; these must have 2, 0, and 4 zeros, respectively. On lines running radially outward to infinity from the vertex, the  $\tilde{W}_\lambda$  are

TABLE I: Numbers of zero lines for solutions corresponding to polyhedra with triangular faces

	$k_F$	$k_E$	$k_V$	$k_{V'}$
$W_+$	1	0	0	-2
$W_r$	-1	1	1	-1
$W_-$	0	0	2	0

the same as the  $W_\lambda$  defined with respect to the origin. On the other hand, on lines running radially into the center from the vertex the roles of  $W_+$  and  $W_-$  are interchanged. Furthermore, the reversal in direction of the outward normal changes the helicity, interchanging zeros and antizeros. The net result is the requirement that

$$\begin{aligned}
 k_V^0 - k_{V'}^0 &= 2, \\
 k_V^1 - k_{V'}^{-1} &= 0, \\
 k_V^{-1} - k_{V'}^1 &= 4.
 \end{aligned} \tag{4.7}$$

These conditions are satisfied by the  $k_V^\lambda$  and  $k_{V'}^\lambda$  given by Eqs. (4.4) and (4.6).

The above arguments uniquely determine the  $k_V^\lambda$  and  $k_{V'}^\lambda$ , but leave some ambiguity as to the  $k_F^\lambda$  and  $k_E^\lambda$ . This ambiguity can be resolved by choosing the solutions of Eq. (4.5) that require the smallest total number of zero lines; this criterion seems energetically reasonable, since it would tend to minimize the angular derivatives of the fields. With this choice, the numbers of zero lines of the various types are given in Table I.

Consider first the lines running through the vertices. On the line running from the vertex out to infinity,  $W_-$  has a double zero and  $W_r$  has a single zero, while on the line connecting the origin and the vertex,  $W_+$  has a double antizero and  $W_r$  again has a single antizero. We can understand these results in terms of energetic arguments. The Higgs zero at the vertex is a source for the magnetic field  $\mathbf{B}$ . From the symmetry of the configuration, we expect the outward magnetic field to have a local maximum, as a function of angle, along the line extending radially outward from the vertex. Recalling Eq. (2.16), we see that local energy density is reduced by minimizing  $|W_-|$  and making  $|W_+|^2$  as close as possible to  $B_r$ .  $W_r$  does not enter explicitly here, but having a zero of  $W_r$  allows a larger value of  $|W_+|$  for a given value of  $|\mathbf{W}|$ , and so is also energetically favored. On the lines running from the vertex

toward the origin, the direction of  $\mathbf{B}$  is inward. This interchanges the roles of  $W_+$  and  $W_-$ . It also reverses the optimum helicity along the zero-lines, so that the zeros on the external lines are replaced by antizeros on the internal ones.

The behavior on the faces and along the edges also makes sense energetically. There is a zero of  $W_+$  and an antizero of  $W_r$  at the center of each face, and a zero of  $W_r$  at the midpoint of every edge. Since these are the points furthest from the zeros of the Higgs field, one would expect  $\phi$  to have a maximum there, which makes it energetically favorable for  $|\mathbf{W}|$  to be small. This is also consistent with the arguments given in Sec. I that the  $J = N - 1$  vector harmonics (which contribute only to  $W_+$ ) should have a zero on each face.

Our discussion has been carried out in a gauge where the Higgs field has a fixed  $SU(2)$  orientation. (This implies the existence of Dirac strings, but these do not affect any of our arguments.) However, one can also work in a nonsingular gauge where the winding number of the asymptotic Higgs field is given by the magnetic charge. Although the Higgs field orientation is a gauge-variant quantity and need not be symmetric, let us consider what a symmetric (or almost symmetric) Higgs field would look like. If we were to ignore the antizero at the center, all of the Higgs winding would have to occur outside the sphere of radius  $R_{\text{cr}}$  on which the Higgs zeros lie. Let us therefore consider the Higgs field on a sphere in this exterior region. As  $N$ , and thus  $R_{\text{cr}}$ , becomes large, a region of fixed transverse size approaches a plane. A symmetric Higgs winding could then be obtained by taking  $\hat{\phi}^a = \phi^a/|\phi| = \delta^{a3}$  above each zero of the Higgs field (i.e., at each of the vertices of the triangulation of the sphere). The lattice dual to this triangulation is a hexagonal one, with the edges of the hexagons being the lines joining the centers of the triangles. To make the total winding number match the magnetic field, the Higgs field should take the opposite orientation,  $\hat{\phi}^a = -\delta^{a3}$  on these hexagonal edges. However, this prescription is exact only in the  $N \rightarrow \infty$  planar limit. With finite  $N$ , and a spherical shape, the Higgs orientation should vary slightly from vertex to vertex. Instead of the above prescription, we should have  $\hat{\phi}^a = \hat{r}^a$  at each of the vertices, with  $\hat{\phi}^a$  taking the opposite orientation along the corresponding hexagon (and the occasional pentagon) of the dual lattice. This variation from vertex to vertex reduces the Higgs winding slightly, so that the total winding number is one less than the number of vertices. This is the origin of the Higgs antizero at the center.

## V. SPHERICAL NON-ABELIAN MAGNETIC BAGS

We now turn to the non-Abelian bag, focusing on the extreme case where all of the Higgs zeros coincide at the origin. For the Abelian bag, the presence of a finite number of Higgs zeros on the bag wall clearly ruled out the possibility of exact spherical symmetry. One might have thought that putting the zeros all at the same point would allow such symmetry for the non-Abelian case, were it not that it was shown long ago [13] that spherical symmetry is only allowed for  $N = 1$ . One way to understand this result is to observe that the components of  $\mathbf{W}$  each have finite numbers of zero lines, whose locations necessarily break the spherical symmetry.<sup>4</sup>

For the Abelian bag, the shape of the bag wall and the polyhedral structure on that wall were given directly by the locations of the Higgs zeros. In the non-Abelian case, the features determining the polyhedral structure are much less obvious. To put this in terms of collective coordinates, recall that the charge  $N$  solution depends on  $4N - 1$  parameters, which can be taken to be the positions and  $U(1)$  phases of the component monopoles, less one overall global phase. For the Abelian bag, even after the positions of the Higgs zeros have been chosen to define its gross structure, there are still roughly  $N$  phase variables that can be adjusted. For the non-Abelian bag, on the other hand, requiring that the Higgs zeros all lie at the origin leaves only  $N - 1$  adjustable parameters, which is far less than would be needed to specify the vertices of an arbitrary polyhedral bag.

Of course, one might ask whether it is even possible to have an arbitrary number of Higgs zeros coincide. The cubic and dodecahedral solutions show that this can be done for  $N = 4$  or 7, and axially symmetric solutions with multiple zeros at the origin can be constructed for arbitrary  $N$ . Note, though, that the latter are toroidal in shape, and for large  $N$  are more disklike than polyhedral. Although we see no reason why polyhedral solutions should not exist for arbitrary large  $N$ , we have no rigorous proof, but must assume that they do.

As with the Abelian bags, we expect the  $2N - 2$  zeros of the lowest,  $J = N - 1$ , harmonics of  $\mathbf{W}$  to lead to a polyhedral shape with  $2N - 2$  faces. In contrast with the Abelian case, where the locations of these faces, including their distances from the center, were uniquely determined by the Higgs zeros, there are now only lines of zeros emanating from the origin.

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<sup>4</sup> The  $N = 1$  monopole solution can be spherically symmetric because  $W_+$  has no zero lines (since  $2N - 2 = 0$ ), while  $W_r$  and  $W_-$  vanish everywhere.

Assuming these lines to be radial, we will have a foliation of space by concentric polyhedra, but with no specific polyhedron uniquely picked out, although the ones lying within the thin wall region in which  $|\mathbf{W}|$  rapidly goes to zero are clearly special.

Recalling Eq. (2.16), we see that it is energetically favorable for  $\mathbf{B}$  to be largest in the directions where  $W_+$  is largest. Since the latter vanishes at the center of each face, it seems likely that the magnetic flux will be greatest along the directions corresponding to the vertices. To spread this flux out as evenly as possible, one would want to maximize the number of vertices for a given value of  $N$ . This is done by requiring that each vertex be trivalent; i.e., that precisely three edges emanate from each vertex. Hence, we expect the non-Abelian polyhedra to be the duals of the triangular polyhedra of the Abelian bags; this is consistent with the cubic and dodecahedral examples. The numbers of vertices, edges, and faces are then given by Eq. (3.4). For large  $N$ , the most symmetric solution would then have surfaces that were mostly covered by approximately regular hexagons, with exactly twelve pentagons distributed among them.<sup>5</sup>

We now turn to the lines of zeros associated with the components of  $\mathbf{W}$ , proceeding as we did for the Abelian case. There will be

- a)  $n_F^\lambda = (2N - 2)\ell_F^\lambda$  lines from the origin, through a face, to infinity ,
- b)  $n_E^\lambda = (6N - 12)\ell_E^\lambda$  lines from the origin, through an edge, to infinity,
- c)  $n_V^\lambda = (4N - 8)\ell_V^\lambda$  lines from a vertex to infinity,

with the  $\ell_a^\lambda$  all integers.

By considering any sphere centered about the origin we obtain the analog of Eq. (4.1),

$$(2N - 2)\ell_F^\lambda + (6N - 12)\ell_E^\lambda + (4N - 8)\ell_V^\lambda = 2N - 2\lambda. \quad (5.1)$$

Proceeding as before, we assume that, at least for large  $N$ , the  $\ell_a^\lambda$  are independent of  $N$ . Equation (5.1) then gives two equations, which are solved by

$$\ell_F^\lambda = 2 - \lambda \quad (5.2)$$

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<sup>5</sup> The topological constraints can also be satisfied by replacing pairs of pentagons by quadrilaterals, or triplets of pentagons by triangles, as in the cube and the tetrahedron.

TABLE II: Numbers of zero lines for solutions corresponding to polyhedra with hexagonal and pentagonal faces

	$\ell_F$	$\ell_E$	$\ell_V$
$W_+$	1	0	0
$W_r$	2	-1	1
$W_-$	3	0	-1

and

$$3\ell_E + 2\ell_V = -1 + \lambda. \quad (5.3)$$

If we again take the solutions that require the fewest zero lines, we obtain the results shown in Table II.

The results for the lines of zeros through the vertices are similar to those for the Abelian case, and are consistent with the magnitude of  $\mathbf{B}$  having a maximum in these directions. The zero of  $W_-$  decreases the right-hand side of Eq. (2.16), while the zero of  $W_r$  allows  $|W_+|^2$  to be larger, again decreasing this contribution to the energy. The multiple zeros of  $W_r$  and  $W_-$  in the faces strongly suggest that  $\phi$  has minima at the centers of the faces.

## VI. CONCLUDING REMARKS

The examples we have studied are not the only possible quasispherical solutions. One could certainly bring together a collection of monopoles to create an Abelian bag with a hexagonal, rather than triangular, lattice. However, this lattice structure would be less regular, with zeros of  $W_+$  on only some faces. Alternatively, one could construct a lattice with more than one monopole at each vertex.

There will also be hybrid bags, with some Higgs zeros in the bag wall and some in the interior. Consider for example, a solution with  $N/2$  zeros arranged symmetrically around a sphere of radius  $R_2 = N$  and  $N/2$  zeros arranged around a sphere of radius  $R_1 < R_2$ . For  $r < R_1$  this would resemble the Abelian bag, but for  $R_1 < r < R_2$  the fields would be similar to those of the non-Abelian bag.

It is unclear, however, what the range of possibilities is for the non-Abelian bag, especially

for the case where the Higgs zeros are all at the origin. Here, the lattice structure emerges from the nonlinear interactions of the fields, rather than being imposed by the location of Higgs zeros at the surface of the bag. As we have already noted, the number of parameters that can be varied is far fewer than would be needed to specify an arbitrary lattice on the bag wall.

There can also be nonspherical bags. One with an Abelian interior can be constructed by assembling an irregular array of separated monopoles and bringing them together just till the point where their cores overlap and  $\phi$  becomes exponentially small in the interior. This requires a nonuniform density of zeros on the bag wall, with the density being highest in the regions where the wall curvature is the greatest. More precisely, the distribution of zeros can be obtained by using the magnetic conductor picture of Bolognesi, and requiring that the monopole density yield a constant dual magnetic potential in the interior of the bag. In fact, it is clear from this approach that, by reducing the monopole density on the bag wall, one can also obtain a nonspherical bag with a nonzero, but approximately uniform, Higgs field in its interior.

Finally, note that as  $N$  and, consequently, the bag radius become large, any finite region of the bag wall can be approximated by a plane, which we may take to be the  $x$ - $y$  plane. In the case of the Abelian bag, the wall would be similar to the monopole wall studied by Ward [2], except that it would contain a triangular, rather than a square, lattice of Higgs zeros. On the side of the wall corresponding to the bag interior (which we choose to be  $z < 0$ ), the magnetic field and the Higgs field would rapidly approach zero with increasing distance from the wall. On the other side, the variation of  $\mathbf{B}$  and  $\phi$  with  $x$  and  $y$  would tend to vanish with distance from the wall. The former would approach a constant vector  $B_i = \delta_{i3}B$ , while asymptotically the latter would increase linearly, with  $\phi \approx Bz$ .

Although  $\mathbf{W}$  would be exponentially small outside the wall, it would not be precisely zero, and so the lines of zeros would persist. The analog of  $W_r$  would be  $W_z$ , while  $W_x \pm iW_y$  would correspond to  $W_{\pm}$ . In the planar limit the constraints on the zeros of all three of these components are all the same. For a triangular lattice with  $E = 3F/2 = 3V$ , the analog of Eq. (4.1) in the region  $0 < z < \infty$ , where there is a nonzero magnetic flux, is

$$2k_F + 3k_E + k_V = 2. \quad (6.1)$$

There is no net magnetic flux through any plane with negative  $z$ , so for  $-\infty < z < 0$  we



have

$$2k_F + 3k_E + k_{V'} = 0, \quad (6.2)$$

where  $k_{V'}$  refers to lines running from  $-\infty$  to a vertex. Because lines of zeros can only end at a zero of the Higgs field,  $k_F$  and  $k_E$  must be continuous in going through the wall. These equations are satisfied by the values in Table I as, of course, they must.

Several possible variations on this picture immediately come to mind. One possibility is to have  $\phi$  asymptotically constant, but nonzero, as  $z \rightarrow \infty$ ; this would be the planar limit of an Abelian bag with radius greater than  $R_{\text{cr}}$ . Corresponding to the case of nested shells of zeros would be a series of parallel planes, each with its own lattice of Higgs zeros and monopole cores. For these solutions,  $\phi$  is approximately linear in  $z$ , but with a slope that changes as one passes through each wall. Finally, there are “monopole sheet” solutions [14, 15] in which  $B$  points outward on both sides of the wall; these cannot be realized as limits of a bag solution.

The case of the non-Abelian bag is perhaps more challenging, because there are no Higgs zeros to fix the location of the wall. Nevertheless, we can understand how a wall can arise by making some simplifying assumptions. To start, let us consider fields that have been averaged over a lattice plaquette in the  $x$  and  $y$  directions, so that they are independent (up to a gauge transformation) of these variables, and only have nontrivial dependence on  $z$ . It follows that  $\mathbf{B}$  is constant and has only a  $z$  component, and hence that we can set  $A_z = 0$ . Let us assume that  $W_z$  also vanishes. Finally, let us assume that the behavior of  $W_x$  and  $W_y$  is similar to that of the lowest monopole harmonics, so that  $W_x = iW_y \equiv w(z)$ .

With these assumptions, Eqs. (2.7) and (2.8) yield two nontrivial equations,

$$\frac{d\phi}{dz} = B - 2|w|^2 \quad (6.3)$$

and

$$\frac{dw}{dz} = -\phi w. \quad (6.4)$$

Differentiating the first of these and then using the original two equations to eliminate  $w$  leads to

$$0 = \frac{d^2\phi}{dz^2} + 2\phi \frac{d\phi}{dz} - 2B\phi. \quad (6.5)$$

In a generic solution of this equation,  $|\phi|$  diverges at both  $-\infty$  and  $\infty$ . However, there

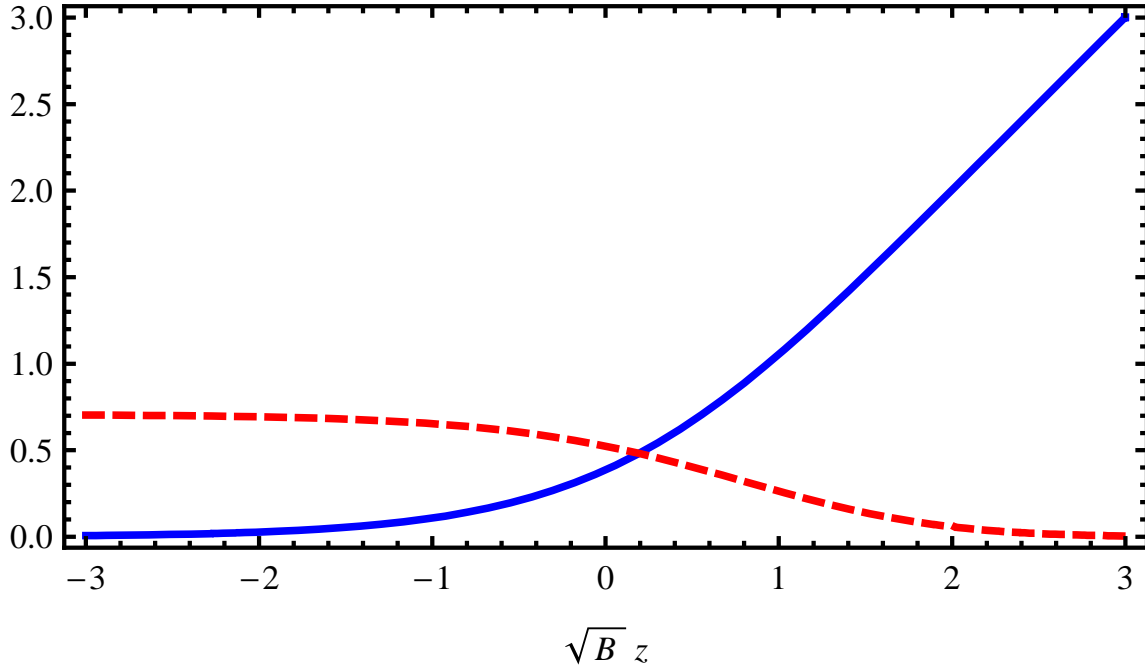


FIG. 1: The functions  $w/\sqrt{B}$  (dashed red line) and  $\phi/\sqrt{B}$  (solid blue line) for the planar solution corresponding to the large  $N$  limit of the non-Abelian bag.

are also solutions with the asymptotic behavior

$$\phi \approx \begin{cases} B(z - z_0), & z \rightarrow \infty, \\ e^{\sqrt{2B}z}, & z \rightarrow -\infty. \end{cases} \quad (6.6)$$

that match our expectations for the wall of the non-Abelian bag; the dependence on the arbitrary parameter  $z_0$  reflects the translation invariance of the theory. Once  $\phi(z)$  is known, Eq. (6.4), together with the fact that  $d\phi/dz$  vanishes at  $z = -\infty$ , gives

$$w(z) = \sqrt{B} \exp \left[ - \int_{-\infty}^z dz' \phi(z') \right]. \quad (6.7)$$

These solutions are shown in Fig. 1. The wall region, in which  $w$  transitions from being essentially constant to being exponentially small, is clearly evident. A rescaling of variables in Eq. (6.5) shows that this has a width of order  $B^{-1/2}$ . This agrees with our estimate, in Sec. III, that the wall of a spherical bag with critical radius should have a thickness  $\sim \sqrt{N}v^{-1} \sim [B(R_{\text{cr}})]^{-1/2}$ .

To sum up, in this paper we have explored BPS monopole solutions in the limit of large magnetic charge. Our results have confirmed and refined the bag picture and elucidated its

fine structure, and have shown that the monopole bag can be realized in a novel non-Abelian manner.

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